

## ON A THEOREM OF ARVANITAKIS

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ABSTRACT. Arvanitakis [2] established recently a theorem which is a common generalization of Michael's convex selection theorem [11] and Dugundji's extension theorem [7]. In this note we provide a short proof of a more general version of Arvanitakis' result.

## 1. INTRODUCTION

Arvanitakis [2] established recently the following result extending both Michael's convex selection theorem [11] and Dugundji's simultaneous extension theorem [7]:

**Theorem 1.1.** [2] *Let  $X$  be a space with property  $c$ ,  $Y$  a complete metric space and  $\Phi: X \rightarrow 2^Y$  a lower semi-continuous set-valued map with non-empty values. Then for every locally convex complete linear space  $E$  there exists a linear operator  $S: C(Y, E) \rightarrow C(X, E)$  such that*

$$(1) \quad S(f)(x) \in \overline{\text{conv}} f(\Phi(x)) \text{ for all } x \in X \text{ and } f \in C(Y, E).$$

*Furthermore,  $S$  is continuous when both  $C(Y, E)$  and  $C(X, E)$  are equipped with the uniform topology or the topology of uniform convergence on compact sets.*

Here,  $C(X, E)$  is the set of all continuous maps from  $X$  into  $E$  (if  $E$  is the real line, we write  $C(X)$ ). We also denote by  $C_b(X, E)$  the bounded functions from  $C(X, E)$ . Recall that a set-valued map  $\Phi: X \rightarrow 2^Y$  is lower semi-continuous if the set  $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$  is open in  $X$  for any open  $U \subset Y$ . A space  $X$  is said to have property  $c$  [2] if  $X$  is paracompact and, for any space  $Y$  and a map  $\phi: X \rightarrow Y$ ,  $\phi$  is continuous if and only if it is continuous on every compact subspace of  $X$ . It is easily seen that the last condition is equivalent to  $X$  being a  $k$ -space (i.e., the topology of  $X$  is determined by its compact subsets, see [8]).

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We provide a short proof of Theorem 1.1. Here is our slightly more general version of Theorem 1.1.

**Theorem 1.2.** *Let  $X$  be a paracompact space,  $Y$  a complete metric space and  $\Phi: X \rightarrow 2^Y$  a lower semi-continuous set-valued map with non-empty values. Then:*

- (i) *For every locally convex complete linear space  $E$  there exists a linear operator  $S_b: C_b(Y, E) \rightarrow C_b(X, E)$  satisfying condition (1) such that  $S_b$  is continuous with respect to the uniform topology and the topology of uniform convergence on compact sets;*
- (ii) *If  $X$  is a  $k$ -space or  $E$  is a Banach space,  $S_b$  can be continuously extended (with respect to both types of topologies) to a linear operator  $S: C(Y, E) \rightarrow C(X, E)$  satisfying (1).*

Our proof of Theorem 1.2 is based on the idea from a result of Repovš, P.Semenov and E.Shchepin [15] that Michael's zero-dimensional selection theorem yields the convex-valued selection theorem.

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## 2. PROOF OF THEOREM 1.2

Let  $E$  be a locally convex linear space. We denote by  $E^*$  the set of all continuous linear functionals on  $E$  with the topology of uniform convergence on the weakly bounded subsets of  $E$ . The second dual  $E^{**}$  is the space of continuous functionals on  $E^*$  with the topology of uniform convergence on the equicontinuous subsets of  $E^*$ . It is well known that the canonical map  $E \rightarrow E^{**}$  is an embedding, see [16].

We need Banach's technique [4] concerning barycenters of some probability measures. First of all, for every compact space  $X$  let  $P(X)$  be the space of all regular probability measures on  $X$  endowed with the  $w^*$ -topology. Each  $\mu \in P(X)$  can also be considered as a continuous linear positive functional on  $C(X)$  (the continuous real-valued functions on  $X$  with the uniform convergence topology) with  $\mu(1_X) = 1$ , where  $1_X$  is the constant function on  $X$  having a value one. Recall that for any  $\mu \in P(X)$  there exists a closed nonempty set  $\text{supp}(\mu) \subset X$  such that  $\mu(g) = \mu(f)$  for any  $f, g \in C(X)$  with  $f|_{\text{supp}(\mu)} = g|_{\text{supp}(\mu)}$ , and  $\text{supp}(\mu)$  is the smallest closed subset of  $X$  with this property. If  $X$  is a Tychonoff space, we consider the following subsets of  $P(\beta X)$ , where  $\beta X$  is the Čech-Stone compactification of  $X$ :

$$P_\beta(X) = \{\mu \in P(\beta X) : \text{supp}(\mu) \subset X\}$$

and

$$\hat{P}(X) = \{\mu \in P(\beta X) : \mu_*(X) = 1\}.$$

Here  $\mu_*(X) = \sup\{\mu(B) : B \subset X \text{ is a Borel subset of } \beta X\}$ . Every map  $h: M \rightarrow E$  generates a map  $P_\beta(h): P_\beta(M) \rightarrow P_\beta(E)$  defined by  $P_\beta(h)(\mu)(\phi) = \mu(\phi \circ h)$ , where  $\mu \in P_\beta(M)$  and  $\phi \in C_b(E)$ . In particular, if  $i_M: M \hookrightarrow E$  is the inclusion of  $M$  into  $E$ , then  $P_\beta(i_M)$  is one-to-one and  $P_\beta(\delta_x) = \delta_x$  for all  $x \in M$  ( $\delta_x$  is the Dirac measure at the point  $x$ ). The functors  $\hat{P}$  and  $P_\beta$  were introduced in [3] and [6], respectively.

Banach [4] defined barycenters of measures from  $\hat{P}(M)$ , where  $M$  is a weakly bounded subset of some locally convex linear space  $E$ . For any such  $M \subset E$  there exists an affine map (called a *barycenter map*)  $b_M: \hat{P}(M) \rightarrow E^{**}$  which is continuous only when  $M$  is bounded in  $E$ , see [4, Theorem 3.2]. A convex subset  $M \subset E$  is called *barycentric* if  $b_M(\hat{P}(M)) \subset M$ . It was established in [4, Proposition 3.10] that any complete bounded convex subset of  $E$  is barycentric. Since for any  $M$  we have  $P_\beta(M) \subset \hat{P}(M)$ , we can apply the Banach arguments with  $\hat{P}(M)$  replaced by  $P_\beta(M)$ , and this is done in the following proposition.

**Proposition 2.1.** *Let  $E$  be a complete locally convex linear space. Then there exists a not necessarily continuous affine map  $b_E: P_\beta(E) \rightarrow E$  such that  $b_E(\mu) \in \overline{\text{conv}}(\text{supp}(\mu))$  for every  $\mu \in P_\beta(E)$ . Moreover, if  $M \subset E$  is a bounded set then the map  $b_E \circ P_\beta(i_M): P_\beta(M) \rightarrow E$  is continuous.*

*Proof.* We follow the arguments from [4]. For every  $\mu \in P_\beta(E)$  we consider the functional  $b_E(\mu): E^* \rightarrow \mathbb{R}$ , defined by  $b_E(\mu)(l) = \mu(l|_{\text{supp}(\mu)})$ ,  $l \in E^*$ .

*Claim.*  $b_E(\mu)$  is continuous for all  $\mu \in P_\beta(E)$ .

Indeed, suppose  $\{l_\alpha\} \subset E^*$  is a net in  $E^*$  converging to some  $l_0 \in E^*$ . This means that  $\{l_\alpha\}$  is uniformly convergent to  $l_0$  on every weakly bounded subset of  $E$ . In particular,  $\{l_\alpha\}$  is uniformly convergent to  $l_0$  on  $\text{supp}(\mu)$ . Consequently,  $\{\mu(l_\alpha)\}$  converges to  $\mu(l_0)$ .

Therefore,  $b_E(\mu) \in E^{**}$  for any  $\mu \in P_\beta(E)$ . On the other hand, since  $\text{supp}(\mu) \subset E$  is compact and  $E$  is complete,  $C(\mu) = \overline{\text{conv}}(\text{supp}(\mu))$  is a compact convex subset of  $E$ . Then, according to [4, Proposition 3.10],  $C(\mu)$  is barycentric and contains  $b_E(\mu)$ . So,  $b_E$  maps  $P_\beta(E)$  into  $E$ . The second half of Proposition 2.1 follows from the fact that  $E$  is embedded in  $E^{**}$  and Theorem 3.2 from [4], which (in our situation) states that the map  $b_E \circ P_\beta(i_M): P_\beta(M) \rightarrow E^{**}$  is continuous provided  $M$  is bounded in  $E$ .  $\square$

The theory of maps between compact spaces admitting averaging operators was developed by Pelczyński [13]. For noncompact spaces we use the following definition [17]: a surjective continuous map  $f: X \rightarrow Y$

admits an averaging operator with compact supports if there exists an embedding  $g: Y \rightarrow P_\beta(X)$  such that  $\text{supp}(g(y)) \subset f^{-1}(y)$  for all  $y \in Y$ . Then the regular linear operator  $u: C_b(X) \rightarrow C_b(Y)$ , defined by

$$(2) \quad u(h)(y) = g(y)(h), \quad h \in C_b(X), \quad y \in Y$$

satisfies  $u(\phi \circ f) = \phi$  for any  $\phi \in C_b(Y)$ . Such an operator  $u$  is called *averaging for  $f$* .

**Proposition 2.2.** *Let  $f: X \rightarrow Y$  be a perfect map admitting an averaging operator with compact supports and  $E$  a complete locally convex linear space. Then there exists a linear operator  $T_b: C_b(X, E) \rightarrow C_b(Y, E)$  such that:*

- (i)  $T_b(h)(y) \in \overline{\text{conv}}(h(f^{-1}(y)))$  for all  $y \in Y$  and  $h \in C_b(X, E)$ ;
- (ii)  $T_b(\phi \circ f) = \phi$  for any  $\phi \in C_b(Y, E)$ ;
- (iii)  $T_b$  is continuous when both  $C_b(X, E)$  and  $C_b(Y, E)$  are equipped with the uniform topology or the topology of uniform convergence on compact sets.

Moreover, if  $Y$  is a  $k$ -space or  $E$  is a Banach space,  $T_b$  can be extended to a linear operator  $T: C(X, E) \rightarrow C(Y, E)$  satisfying conditions (i) – (iii) with  $C_b(X, E)$  and  $C_b(Y, E)$  replaced, respectively, by  $C(X, E)$  and  $C(Y, E)$ .

*Proof.* A similar statement to the first part was proved in [17, Proposition 3.1]. We fix an embedding  $g: Y \rightarrow P_\beta(X)$  with  $\text{supp}(g(y)) \subset f^{-1}(y)$ ,  $y \in Y$ . For every  $h \in C_b(X, E)$  consider the map

$$(3) \quad T_b(h): Y \rightarrow E, \quad T_b(h)(y) = b_E(P_\beta(i_{h(X)})(\nu_y)),$$

where  $i_{h(X)}: h(X) \hookrightarrow E$  is the inclusion and  $\nu_y \in P_\beta(h(X))$  is the measure  $P_\beta(h)(g(y))$ . According to Proposition 2.1,  $T_b(h)$  is continuous (recall that  $h(X) \subset E$  is bounded). It also follows from the definition of the map  $b_E$  that  $T_b$  is linear. Since  $\text{supp}(g(y)) \subset f^{-1}(y)$  and  $\text{supp}(P_\beta(i_{h(X)})(\nu_y)) \subset h(f^{-1}(y))$ ,  $y \in Y$ , we have  $b_E(P_\beta(i_{h(X)})(\nu_y)) \subset \overline{\text{conv}}(h(f^{-1}(y)))$  (see Proposition 2.1). So,  $T_b$  satisfies condition (i). Moreover,  $T_b(h)$  belongs to  $C_b(Y, E)$  because  $T_b(h)(y) \in \overline{\text{conv}}(h(X))$  for all  $y \in Y$ . It follows directly from (2) and (3) that  $T_b$  satisfies condition (ii). To prove (iii), assume  $K \subset Y$  is compact and let  $W_1 = \{\phi \in C_b(Y, E) : \phi(K) \subset V_1\}$ , where  $V_1$  is a convex neighborhood of 0 in  $E$ . Obviously,  $W_1$  is a neighborhood of the zero function in  $C_b(Y, E)$ . Take a convex neighborhood  $V_2$  of 0 in  $E$  with  $\overline{V_2} \subset V_1$  and let  $W_2 = \{h \in C_b(X, E) : h(H) \subset V_2\}$ ,  $H = f^{-1}(K)$ . Since  $H$  is compact (recall that  $f$  is a perfect map),  $W_2$  is a neighborhood of 0 in  $C_b(X, E)$ . Moreover, for all  $y \in Y$  and  $h \in W_2$  we have  $T_b(h)(y) \in \overline{\text{conv}}(h(H)) \subset \overline{V_2} \subset V_1$ . So,  $T_b(W_2) \subset W_1$ . This provides

continuity of  $T_b$  with respect to the topology of uniform convergence on compact sets. Similarly, one can show that  $T_b$  is also continuous with respect to the uniform topology.

Assume that  $Y$  is a  $k$ -space and  $h \in C(X, E)$ . Then formula (3) provides a map  $T(h): Y \rightarrow E$  satisfying conditions (i) and (ii). We need to show that  $T(h)$  is continuous on every compact set  $L \subset Y$ . And this follows from Proposition 2.1 because the set  $h(f^{-1}(L)) \subset E$  is compact. So,  $T(h)$  is continuous and, obviously,  $T(h) = T_b(h)$  for all  $h \in C_b(X, E)$ . Continuity of  $T$  follows from the same arguments we used to prove continuity of  $T_b$ .

If  $E$  is a Banach space, then every  $T(h)$ ,  $h \in C(Y, E)$ , is continuous without the requirement  $Y$  to be a  $k$ -space. Indeed, we fix  $y_0 \in Y$  and  $h \in C(X, E)$ . Let  $V$  be a bounded closed neighborhood of  $h(f^{-1}(y_0))$  in  $E$ . Then  $h^{-1}(V)$  is a neighborhood of  $f^{-1}(y_0)$  and, since  $f$  is a perfect map, there exists a closed neighborhood  $U$  of  $y_0$  in  $Y$  with  $W = f^{-1}(U) \subset h^{-1}(V)$ . Then, according to Proposition 2.1, the map  $b_E \circ P_\beta(i_V): P_\beta(V) \rightarrow E$  is continuous. On the other hand  $P_\beta(h)$  maps continuously  $P_\beta(W)$  into  $P_\beta(V)$  and  $g(U) \subset P_\beta(W)$  is homeomorphic to  $U$  (recall that  $g$  is an embedding of  $Y$  into  $P_\beta(X)$ ). Hence,  $T(h)$  is continuous on  $U$ . Because  $U$  is a neighborhood of  $y_0$  in  $Y$ , this implies continuity of  $T(h)$  at  $y_0$ .  $\square$

*Proof of Theorem 1.2.* Suppose  $X$ ,  $Y$ ,  $\Phi$  and  $E$  satisfy the hypotheses of Theorem 1.2. By [15] (see also [14]), there exists a zero-dimensional paracompact space  $X_0$  and a perfect surjection  $f: X_0 \rightarrow X$  admitting a regular averaging operator. By Proposition 2.2, there exists a linear operator  $T_b: C_b(X_0, E) \rightarrow C_b(X, E)$  satisfying conditions (i) – (iii). The map  $\tilde{\Phi}: X_0 \rightarrow 2^Y$ ,  $\tilde{\Phi}(z) = \overline{\Phi(f(z))}$ , is lower semi-continuous with closed non-empty values in  $Y$ . So, according to the Michael's 0-dimensional selection theorem [12],  $\tilde{\Phi}$  has a continuous selection  $\theta: X_0 \rightarrow Y$ . Now, we define the linear operator  $S_b: C_b(Y, E) \rightarrow C_b(X, E)$  by  $S_b(h) = T_b(h \circ \theta)$ ,  $h \in C_b(Y, E)$ . Obviously,  $\theta(f^{-1}(x)) \subset \overline{\Phi(x)}$  for every  $x \in X$ . Then, according to (i), for all  $h \in C_b(Y, E)$  and  $x \in X$  we obtain

$$S_b(h)(x) = T_b(h \circ \theta)(x) \subset \overline{\text{conv}}((h \circ \theta)(f^{-1}(x))) \subset \overline{\text{conv}}(h(\Phi(x))).$$

Continuity of  $S_b$  follows from continuity of  $T_b$  and the map  $\theta$ .

If  $X$  is a  $k$ -space or  $E$  is a Banach space, the operator  $T_b$  can be extended to a linear operator  $T: C(X_0, E) \rightarrow C(X, E)$  satisfying conditions (i) – (iii) from Proposition 2.2. Then  $S: C(Y, E) \rightarrow C(X, E)$ ,  $S(h) = T(h \circ \theta)$ , is the required linear operator extending  $S_b$ .

### 3. REMARKS

Let us show first that Theorem 1.2 implies Michael's selection theorem. Assume  $X$  is paracompact,  $Y$  is a Banach space and  $\Phi: X \rightarrow 2^Y$  a lower semi-continuous map with closed convex values. Then, by Theorem 1.2 there exists a linear operator  $S: C(Y, Y) \rightarrow C(X, Y)$  satisfying condition (1). Since the values of  $\Phi$  are convex and closed, condition (1) yields that  $S(id_Y)(x) \in \Phi(x)$  for all  $x \in X$ , where  $id_Y$  is the identity on  $Y$ . Hence,  $S(id_Y)$  is a continuous selection for  $\Phi$ .

The original Dugundji theorem [7] states that if  $X$  is a metric space,  $A \subset X$  its closed subset and  $E$  a locally convex linear space, then there exists a linear operator  $S: C(A, E) \rightarrow C(X, E)$  such that  $S(f)$  extends  $f$  for any  $f \in C(A, E)$ . When both  $E$  and  $A$  are complete, Dugundji theorem can be derived from Theorem 1.2. Indeed, let  $A$  be a completely metrizable closed subset of a paracompact  $k$ -space  $X$  and  $E$  a complete locally convex linear space. Consider the set-valued map  $\Phi: X \rightarrow 2^A$ ,  $\Phi(x) = \{x\}$  if  $x \in A$  and  $\Phi(x) = A$  if  $x \notin A$ . Let  $S: C(A, E) \rightarrow C(X, E)$  be a linear operator satisfying (1). Then  $S(f)(x) = f(x)$  for all  $f \in C(A, E)$  and  $x \in A$ . So,  $S$  is an extension operator. If  $X$  is not necessarily a  $k$ -space, there exists an extension linear operator  $S_b: C_b(A, E) \rightarrow C_b(X, E)$ .

Heath and Lutzer [10, Example 3.3] provided an example of a paracompact  $X$  and a closed set  $A \subset X$  homeomorphic to the rational numbers such that there is no extension operator from  $C(A)$  to  $C(X)$ . This space is the Michael's line, i.e., the real line with topology consisting of all sets of the form  $U \cup V$ , where  $U$  is an open subset of the rational numbers and  $V$  is a subset of the irrational numbers. It is easily seen that this a  $k$ -space. So, the assumption in the above result  $A$  to be completely metrizable is essential.

The original Dugundji theorem with  $E$  complete can be derived from Proposition 2.2 and the well known fact that every closed subset of a zero-dimensional metric space  $X$  is a retract of  $X$ , see for example [9, Problem 4.1.G]. Indeed, assume  $X$  is a metric space and  $A \subset X$  its closed subset. By [6], there exists a zero-dimensional metric space  $X_0$  and a perfect surjection  $f: X_0 \rightarrow X$  admitting an averaging operator. Let  $A_0 = f^{-1}(A)$  and  $r: X_0 \rightarrow A_0$  be a retraction. Define the linear operator  $S: C(A, E) \rightarrow C(X, E)$  by  $S(h) = T(h \circ f \circ r)$ , where  $E$  is a complete locally convex linear space,  $h \in C(A, E)$  and  $T: C(X_0, E) \rightarrow C(X, E)$  is the operator from Proposition 2.2. It follows from Proposition 2.2(i) that  $S$  is an extension operator.

The proof of Theorem 1.2 is based on two main facts: the 0-dimensional Michael's selection theorem and the Repovš-Semenov-Shchepin result

[15] that each paracompactum is a continuous image of under a perfect map admitting an averaging operator. So, the 0-dimensional Michael's selection theorem implies not only the convex-valued section theorem, but it also implies the Dugundji extension theorem. Actually we have the following corollary from Proposition 2.2 ( $\text{Sel}(\Phi)$  denotes all continuous selections for  $\Phi$ ).

**Corollary 3.1.** *Let  $f: X \rightarrow Y$  be a perfect map admitting an averaging operator with compact supports and  $E$  a Banach space. Suppose  $\Phi: Y \rightarrow 2^E$  is a lower semi-continuous set-valued map with closed convex non-empty values. Then there exists an affine map from  $\text{Sel}(\Phi \circ f)$  to  $\text{Sel}(\Phi)$  which is continuous when both  $\text{Sel}(\Phi \circ f)$  and  $\text{Sel}(\Phi)$  are equipped with the uniform topology or the topology of uniform convergence on compact sets.*

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